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BESSEL TYPE TRANSFORM AND RELATED RESULTS

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Abstract: In this paper we prove the estimates for the Bessel transform in the space $L^2_p(\mathbb{R}_+)$ on certain classes of functions by using a Bessel type generalized translation.

Keywords and Phrases: Bessel type operator, Bessel type transform, Bessel type generalized translation.

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1. Introduction and Preliminaries

Integral transforms play a vital role in diverse research area such as in engineering Mathematics and Mathematical physics. We prepare this paper by motivation of the work in [5]. Titchmarsh [8] characterized the set of functions $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition for the Fourier transform namely, we have

Theorem 1.1. *Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalent:*

$$(i)\ \|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O(h^\alpha) \text{ as } h \to 0$$

$$(ii) \int_{|\lambda|>r} |\mathfrak{F}(\lambda)|^2 d\lambda = O(r^{-2r}) \text{ as } r \to \infty,$$

where $\mathfrak{F}$ stands for the Fourier transform of f.

Our main objective of this paper is to establish a generalization of Theorem 1.1 in the Bessel type transform setting by means of the Bessel type generalized translation. We point out that similar results have been established in the context of noncompact rank 1 Riemannian symmetric spaces and of Jacobi transform (see [2, 7]).

In this section we give some definitions and preliminaries concerning the Bessel type transform. Throughout this paper, $a-b$ is assumed to be a positive real number,

Let $D = D_x^2 + \frac{a-b}{x}D_x,\ D_x = \frac{d}{dx}$ be the Bessel type differential operator. Now we introduce the normalized Bessel function of the first kind $j_{\frac{a-b-1}{2}}$ defined by

$$j_{\frac{a-b-1}{2}}(z) := \Gamma\left(\frac{a-b+1}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma\left(n+\frac{a-b-1}{2}\right)}\left(\frac{z}{2}\right)^{2n}, (a-b>0, z\in\mathbb{C}). \quad (1.1)$$

where $\Gamma(x)$ is the gamma function (see [4]). Then we find this function $y(z) := j_{\frac{a-b-1}{2}}(z)$ would satisfy the initial conditions $y(0) = 1$ and $y'(0) = 0$, and is an entire function (of course, infinitely differentiable at each point $z \in \mathbb{C}$).
Now from (1.1), we have

$$\lim_{z\to 0} = \frac{j_{a-b-1}(z)-1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow |j_{\frac{a-b-1}{2}}(z) - 1| \geq c|z|^2. \quad (1.2)$$

From [1], we can see that

$$|j_{\frac{a-b-1}{2}}(x)| \leq 1 \quad (1.3)$$

and

$$1 - j_{\frac{a-b-1}{2}}(x) = O(x^2), 0 \leq x \leq 1. \quad (1.4)$$

Assume that $L^2_{\frac{a-b-1}{2}}(\mathbb{R}_+),\ (a-b) \geq 0$ is the Hilbert space of measurable functions f(x) on $\mathbb{R}_+$ with the finite norm

$$\|f\| = \|f\|_{2,a,b} = \left(\int_0^{\infty}|f(x)|^2 x^{a-b}dx\right)^{1/2}.$$

Given $f \in L^2_{\frac{a-b-1}{2}}(\mathbb{R}_+)$, the Bessel type transform is defined by

$$\hat{f}(\lambda) = \int_0^{\infty} f(t) j_{\frac{a-b-1}{2}}(\lambda t) t^{a-b} dt, \lambda \in \mathbb{R}_+.$$

The inverse Bessel type transform is given by the formula

$$f(t) = \left(2^{\frac{a-b-1}{2}}\Gamma\left(\frac{a-b+1}{2}\right)\right)^{-2}\int_0^\infty \hat{f}(\lambda)j_{\frac{a-b-1}{2}}(\lambda t)\lambda^{a-b}d\lambda.$$

We have the Parseval's identity (see [3])

$$\int_0^\infty |\hat{f}(\lambda)|^2\lambda^{a-b}d\lambda = 2^{a-b-1}\Gamma^2(\frac{a-b+1}{2})\int_0^\infty |f(t)|^2 t^{a-b}dt.$$

In $L^2_{\frac{a-b-1}{2}}(\mathbb{R}_+)$, consider the Bessel type generalized translation T_h (see [3, p.121])

$$T_h f(x) = c_{\frac{a-b-1}{2}}\int_0^\pi f(\sqrt{x^2+h^2-2xh\cos t})\sin^{2p} t, (a-b)\geq 0, h>0$$

where

$$c_{\frac{a-b-1}{2}} = \left(\int_0^\pi \sin^{a-b-1} t dt\right)^{-1} = \frac{\Gamma(\frac{a-b+1}{2})}{\sqrt{\pi}\Gamma(\frac{a-b}{2})}.$$

We note from [6] the important properties of Bessel type transform

$$(\widehat{Df})(\lambda) = (-\lambda^2)\hat{f}(\lambda) \tag{1.5}$$

and

$$(\hat{T_h}f)(\lambda) = j_{\frac{a-b-1}{2}}(\lambda h)\hat{f}(\lambda). \tag{1.6}$$

We now define the differences of first and higher orders as

$$\Delta_h f(x) = T_h f(x) - f(x) = (T_h - E)f(x)$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T_h - E)^k f(x) = \sum_{i=1}^{\infty}(-1)^{k-i}\begin{pmatrix} k \\ i \end{pmatrix} T_h^i f(x), \tag{1.7}$$

where $T_h^0 f(x) = f(x), T_h^i f(x) = T_h(T_h^{i-1}f(x)), i = 1,2,\ldots,k;\; k = 1,2,\ldots$ and E is the unit operator in the space $L^2_p(\mathbb{R}_+)$.

2. Main Results

We need following Lemma.

Lemma 2.1. *For $f \in L^2_p(\mathbb{R}_+)$,*

$$\|\Delta_h^k D^r f(x)\|^2 = \int_0^\infty t^{4r}|j_{\frac{a-b-1}{2}}(th) - 1|^{2k}|\hat{f}(t)|^2 t^{a-b}dt.$$

Proof. We have from (1.5) that

$$(\hat{D^r} f)(t) = (-1)^r t^{2r} \hat{f}(t);\ r = 0, 1, 2 \dots \tag{2.1}$$

Using formulae (1.6) and (2.1), we obtain

$$T_h^i \hat{D}^r f(t) = (-1)^r j^i_{\frac{a-b-1}{2}} t^{2r} \hat{f}(t); 1 \leq i \leq k. \tag{2.2}$$

From formulas (1.7) and (2.2) the image $\Delta_h^k D^r f(x)$ under Bessel type transform has the form

$$T_h^k \hat{D}^r f(t) = (-1)^r (j_{\frac{a-b-1}{2}}(th) - 1)^k t^{2r} \hat{f}(t).$$

Result follows from the Parseval's identity.
Thus proof is completed.

Theorem 2.2. *Let $f \in L_p^2(\mathbb{R}_+)$. Then the following are equivalents*

$$(i) \|\Delta_h^k D^r f(x)\| = O(h^\alpha) \ as\ h \to 0, (0 < \alpha < k)$$

$$(ii) \int_s^\infty t^{4r} |\hat{f}(t)|^2 t^{a-b} dt = O(s^{-2\alpha}) \ as\ s \to +\infty.$$

Proof. (i)⇒(ii)

Suppose that $\|\Delta_h^k D^r f(x)\| = O(h^\alpha)$ as $h \to 0$.
From Lemma 2.1, we have

$$\|\Delta_h^k D^r f(x)\|^2 = \int_0^\infty t^{4r} |j_{\frac{a-b-1}{2}}(th) - 1|^{2k} |\hat{f}(t)|^2 t^{a-b} dt.$$

From(1.2), we obtain

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |j_{\frac{a-b-1}{2}}(th) - 1|^{2k} |\hat{f}(t)|^2 t^{a-b} dt \geq \frac{c^{2k} \eta^{4k}}{2^{4k}} \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\hat{f}(t)|^2 t^{a-b} dt.$$

Now there exist a positive constant C such that

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} t^{4r} |\hat{f}(t)|^2 t^{a-b} dt \leq C \int_0^\infty t^{4r} |j_{\frac{a-b-1}{2}}(th) - 1|^{2k} |\hat{f}(t)|^2 t^{a-b} dt \leq C h^{2\alpha}.$$

Thus, we have

$$\int_s^{2s} t^{4r} |\hat{f}(t)|^2 t^{a-b} dt = O(s^{-2\alpha}),\ for\ all\ s > 0.$$

Moreover, we have

$$\begin{aligned}\int_s^{\infty} t^{4r}|\hat{f}(t)|^2 t^{a-b}dt &= \sum_{j=0}^{\infty}\int_{2^j s}^{2^{j+1}s} t^{4r}|\hat{f}(t)|^2 t^{a-b}dt\\ &\le C\sum_{j=0}^{\infty}(2^j s)^{-2\alpha}\\ &\le Cs^{-2\alpha}.\end{aligned}$$

This proves that

$$\int_0^{\infty} t^{4r}|\hat{f}(t)|^2 t^{a-b}dt = O(s^{-2\alpha}),\ as\ s\to+\infty.$$

(ii) $\Rightarrow$ (i).
Suppose that $\int_0^{\infty} t^{4r}|\hat{f}(t)|^2 t^{a-b}dt = O(s^{-2\alpha}),\ as\ s\to+\infty.$

Now we have to show that

$$\int_0^{\infty} t^{4r}|j_{\frac{a-b-1}{2}}(th)-1|^{2k}|\hat{f}(t)|^2 t^{a-b}dt = O(h^{2\alpha}),\ as\ h\to 0.$$

We write

$$\int_0^{\infty} t^{4r}|j_{\frac{a-b-1}{2}}(th)-1|^{2k}|\hat{f}(t)|^2 t^{a-b}dt = I_1+I_2,$$

where

$$I_1 = \int_0^{1/h} t^{4r}|j_{\frac{a-b-1}{2}}(th)-1|^{2k}|\hat{f}(t)|^2 t^{a-b}dt$$

and

$$I_2 = \int_{1/h}^{\infty} t^{4r}|j_{\frac{a-b-1}{2}}(th)-1|^{2k}|\hat{f}(t)|^2 t^{a-b}dt.$$

Now by using formula (1.3), we obtain

$$I_2 \le 4^k\int_{1/h}^{\infty} t^{4r}|\hat{f}(t)|^2 t^{a-b}dt = O(h^{2\alpha}),\ as\ h\to 0.$$

Now, set

$$\psi(t) = \int_1^{\infty} x^{4r}|\hat{f}(x)|^2 x^{a-b}dx.$$

By formula (1.4) and integration by parts, we have,

$$\begin{aligned}
I_1 &= -\int_0^{1/h} |j_{\frac{a-b-1}{2}}(th) - 1|^{2k} |\psi'(t)| dt \\
&\leq -h^{2k} \int_0^{1/h} t^{2k} \psi'(t) dt \\
&\leq -\psi(\frac{1}{h}) + 2kh^{2k} \int_0^{1/h} t^{2k-1-2\alpha} dt.
\end{aligned}$$

we see that for $\alpha < k$, the integral exists. Thus we have

$$\begin{aligned}
I_1 &\leq \frac{2k}{2k-2\alpha} h^{2k} h^{-2k+2\alpha} \\
&\leq ch^{2\alpha}.
\end{aligned}$$

Thus the proof is completed.

Corollary 2.3. *Let $f \in L_p^2(\mathbb{R}_+), ((a-b) \geq 0)$, and let*

$$\|\Delta_h^k D^r f(x)\| = O(h^\alpha), \text{ as } h \to o.$$

Then

$$\int_s^\infty |\hat{f}(t)|^2 t^{a-b} dt = O(s^{-4r-2\alpha}) \text{ as } s \to +\infty.$$

Remarks:

(i). If we take $a = p + \frac{3}{4}$, $b = -p - \frac{1}{4}$ throughout this paper then we get the results studied in [5].

(ii). Author claims that results studied in this paper are more general than that of [5].

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